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# Algebraic method in game theory

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**Abstract**

In this paper we apply algebraic methods to game theory. The central objects of study in game theory, Nash equilibria, can be characterized in several ways. We focus on their characterization as the solutions to certain systems of polynomial equations and inequalities. Thus we bring to bear the techniques of commutative algebra, algebraic geometry, and combinatory used in solving polynomial systems.

**Keywords:** algebraic methods, game theory, nash equilibria, solutions, systems, equations, techniques, algebra, geometry, combinatory

## Introduction

A graph is given by an arrangement of components, called hubs or vertices, together with an arrangement of pairs of hubs, called edges. We will just consider finite graphs, that is, graphs with a limited arrangement of vertices. In the event that the edges are requested pairs, we say the graph is coordinated, and each edge goes from its first element (or source, or tail) to its second element (or target, or head). On the off chance that the edges are unordered sets (i.e., sets of two hubs), we say the graph is undirected, and each edge goes between its components. We will just consider directed graphs. An edge from a hub to itself is known as a self-circle. The graphs we will consider will all be straightforward, i.e., they will have no self-circles and won't have different edges in a similar heading between the same pair of nodes. A way in a chart is an arrangement $v_0, v_1, \dots, v_n$ of nodes such that $(v_i, v_{i+1})$ is an edge in the graph for each $i = 0, 1, \dots, n-1$. We say the path goes from $v_0$ to $v_n$ or between $v_0$ and $v_n$.

A rooted tree is a unique sort of coordinated graph. It has a recognized hub called the root, and its edges are called branches. No edge goes to the root. Each other node v has a one of a kind edge going to it. The hub w at the tail of this edge is called v's parent, and v is said to be w's tyke. For every hub v, there is an interesting way in the tree from the root to v. We can find this way by moving in reverse and finding v's parent, then the parent of its parent, etc, until we achieve the hub with no parent the root. A hub without any youngsters is known as a leaf. On the off chance that there is a (coordinated) way in the tree from a hub v to a hub w, we say v is a predecessor of w and w is a relative of v. Each node is a progenitor and a relative of itself, and furthermore a relative of the root. For every hub v in a tree, the sub tree underneath v has the relatives of v as vertices and an indistinguishable edge between these vertices from were in the original tree. The hub v itself is the base of this sub tree.

## Review of literature

A polytope is the arrangement of answers for some arrangement of no strict straight imbalances in Euclidean space. In the event that this framework is irredundant (that is, no imbalances can be dropped without changing the polytope) and there are focuses in the polytope which fulfill one of the disparities entirely, then the arrangement of focuses in the polytope where it holds with balance is known as an aspect of the polytope. One can consider features of the aspects, features of the aspects of the features, etc; these are called faces. Each such dropping chain of appearances, where each face is an aspect of the past one, in the end ends in a face which is a solitary point, or vertex.

Given a finite set of points $v_0, \dots, v_d$ in a real vector space such that $v_1 - v_0, \dots, v_d - v_0$ are linearly independent, the simplex with those points as vertices is the set

$$\{\textstyle\sum_{i=0}^{n} \lambda_i v_i : 0 < \lambda_i \in \mathbb{R} \text{ for } i = 0,1,\dots,n \text{ and } \sum_{i=0}^{n} \lambda_i = 1\}$$

The (d-dimensional) possibility simplex over a fixed set of measures $v_0, \dots, v_d$ is achieved by classifying $v_0$ with the origin and $v_1, \dots, v_d$ with the unit coordinate vectors in $\mathbb{R}^d$. It is the polytope $\{(\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d : \lambda_i \geq 0$ for all $i$ and $\sum_{i=1}^{d} \lambda_i \leq 1\}$.

The notion of the determinant of a matrix M is commonplace from linear algebra: it is the total of certain marked results of passages of M. We will likewise require the thought of the perpetual of M. It is the entirety of similar results of sections of M, yet without the extra sign factors.

The set of Nash equilibria for a game with nonexclusive result capacities is limited. This infers the arrangement of completely blended Nash equilibria for a game with nonexclusive result capacities is likewise limited. These are the genuine answers for an arrangement of polynomial conditions and imbalances. The complex answers for the framework of equations are called quasiequilibria. In this way, the arrangement of completely blended Nash equilibria is a sub set of the arrangement of quasiequilibria. Actually, the arrangement of quasiequilibria is additionally limited in the most nonspecific case, and its cardinality can be processed as an element of the numbers of immaculate procedures of the players. Subsequently, this is an upper bound on the quantity of completely blended Nash equilibria. Indeed, even in a no nonexclusive case, the length of the arrangement of quasiequilibria is limited; its cardinality will be limited above by the number in the generic case.

**Generic number of quasiequilibria**

McKelvey and McLennan have figured the correct number of quasiequilibria for games in the most nonspecific case. The accompanying theorem generalizes the circumstance in which the result lattices have more structure.

**Theorem 1-** Supposing that $0 < d \in \mathbb{N}$ and those we are given a partition $\{1, ...., d\} = \coprod_{i=1}^{N} \mathrm{T}_i, \mathrm{f}(1, ...., d)$, write $d_i = |\mathrm{T}_i|$. Supposing further that we are given a directed graph G, the polynomial graph, on d vertices, denoted $\nu_1, \cdots, \nu_d$, without self-loops and with the property that for any $\nu_j$ and $\mathrm{T}_i, \mathrm{f}$ there is some $k \in \mathrm{T}_i$ such that there is an edge from $\nu_j$ to $\nu_k$ in G, then for every $k \in \mathrm{T}_i$ there is an edge from $\nu_j$ to $\nu_k$ in G. Let

$$\begin{aligned} \mathrm{f}_1(\sigma_1, ...., \sigma_d) &= 0 \\ \mathrm{f}_2(\sigma_1, ...., \sigma_d) &= 0 \\ &\vdots \\ \mathrm{f}_d(\sigma_1, ...., \sigma_d) &= 0 \end{aligned}$$

Be a system (1) of d polynomial equations in d variables $\sigma_1, \cdots, \sigma_d$ with the following properties:

1. All monomials occurring in the $\mathrm{f}_i$'s area four-sided free.
2. If $\sigma_j, \sigma_k \in \mathrm{T}_i$ with $j \neq k$ then $\sigma_j$ and $\sigma_k$ do not both happen in any monomial of any of the $\mathrm{f}_i$'s.
3. If there is no edge from $\nu_j$ to $\nu_k$ in G then the variable $\sigma_k$ does not occur in $\mathrm{f}_j$.

Thus, the equations are multi-linear, and they are linear over the variables from each $\mathrm{T}_i$. Construct a $d \times d$ matrix M as follows: If variable $\mathrm{f}_j$ occurs in the polynomial $\mathrm{f}_i$, with $\mathrm{T}_i$ the subset containing $\nu_k$, then

$$\mathrm{M}_{jk} = \frac{1}{(d_i!)^{1/d_i}},$$

Otherwise $\mathrm{M}_{jk} = 0$, if the system is 0-dimensional then the number of its solutions in $(\mathbb{C}^*)^d$ (i.e. such that $\sigma_k \neq 0$ for all *k*) is bounded above by the permanent of M, and is equal to the permanent of M for generic coefficients.

**Proof-** Without loss of generality, assume

$$\mathrm{T}_i = \left\{1 + \sum_{l=1}^{i-1} d_l,\ 2 + \sum_{l=1}^{i-1} d_l, ...., d_i + \sum_{l=1}^{i-1} d_l\right\}.$$

That is, that they $\mathrm{T}_i$'s are contiguous.

Let $a_{ij} = 1$ if there is an edge in G from $\nu_j$ to $\nu_k$ for $k \in \mathrm{T}_i$, and $a_{ij} = 0$ otherwise. Then the Newton polytope $\mathrm{P}_j$ of $\mathrm{f}_j$ is the Cartesian product $\mathrm{P}_{1j} \times \mathrm{P}_{2j} \times .... \times \mathrm{P}_{Nj}$, where $\mathrm{P}_{ij}$ is the convex hull of the scaled coordinate vectors $\{a_{ij} e_k | \ k \in \mathrm{T}_i\}$ and the origin. For $i$ with $a_{ij} = 1$, $\mathrm{P}_{ij}$ is the $d_i$-dimensional unit simplex, and for $i$ with $a_{ij} = 0$, $\mathrm{P}_{ij}$ reprobates to the $d_i$-dimensional origin (which is a 0-dimensional simplex). By the Bernstein-Kouchnirenko Theorem, it suffices to show that the mixed volume of the polytopes $\mathrm{P}_1, ...., \mathrm{P}_d$ is given by the permanent of M. Let

Let $\mathrm{Q}_j = \lambda_1 \mathrm{P}_1 + \cdots + \lambda_j \mathrm{P}_j$, where + denotes Minkowski addition and the scale factors $\lambda_1, ...., \lambda_j$ are parameters. We show by induction on $j$ that $\mathrm{Q}_j = \mathrm{Q}_{1j} \times \mathrm{Q}_{2j} \times .... \times \mathrm{Q}_{Nj}$, where $\mathrm{Q}_{ij}$ is the convex hull and the origin of

$$\{(a_{i1}\lambda_1 + a_{i2}\lambda_2 + \cdots + a_{ij}\lambda_j) e_k | \ k \in \mathrm{T}_i\}.$$

(If $a_{i1}\lambda_1 + a_{i2}\lambda_2 + \cdots + a_{ij}\lambda_j = 0$, Then $\mathrm{Q}_{ij}$ degenerates to the origin). The base case follows from our characterization of $\mathrm{P}_j$ above. Now consider the Minkowski sum of $\mathrm{Q}_j = \mathrm{Q}_{1j} \times \mathrm{Q}_{2j} \times .... \times \mathrm{Q}_{Nj}$ and $\lambda_{j+1}\mathrm{P}_{j+1} = (\lambda_{j+1}\mathrm{P}_{1(j+1)} \times ... \times \lambda_{j+1}\mathrm{P}_{N(j+1)})$, it follows from the definition of Minkowski sum that this is

$(Q_{1j} + \lambda_{j+1}P_{1(j+1)}) \times \ldots \times (Q_{Nj} + \lambda_{j+1}P_{N(j+1)})$ and (using the induction hypothesis) that each factor $Q_{ij} + \lambda_{j+1}P_{i(j+1)}$ is equal to the convex hull of and the origin.

$$\{(a_{i1}\lambda_1 + a_{i2}\lambda_2 + \cdots + a_{ij}\lambda_j + a_{i(j+1)}e_{j+1})e_k \mid k \in T_i\}$$

The $d_i$-dimensional volume of the $d_i$-dimensional unit simplex scaled by $\lambda$ in each dimension is

$$\frac{\lambda^{d_i}}{(d_i)!}$$

We are involved in the $d$-dimensional volume of $Q_d$. If $a_{i1} = a_{i2} = \cdots = a_{id} = 0$ for some $i$, then this volume disappears, and hence the mixed volume also vanishes. In this case the *k*th column of the matrix M will be all zeroes for any $k \in T_i$, so the permanent of M also vanishes, and the theorem holds. So assume that for each $i$, there is some $j$ with $a_{ij} = 1$. Then the volume of $Q_d$ is

$$\prod_{i=1}^{N} \frac{(a_{i1}\lambda_1 + \cdots + a_{id}\lambda_d)^{d_i}}{d_i!}$$

Let $(g_{jk})$ be the adjacency matrix of G, that is, $g_{jk} = 1$ if there is an edge in G from $v_j$ to $v_k$ and $g_{jk} = 0$ otherwise. Then $a_{ij} = g_{jk}$ for all $k \in T_i$, so the volume of $Q_d$ is

$$\frac{\prod_{k=1}^{d}(g_{1k}\lambda_1 + \cdots + g_{dk}\lambda_d)}{\prod_{i=1}^{N} d_i!}$$

The mixed volume of $P_1, \ldots, P_d$ is the coefficient of $\lambda_1, \lambda_2, \ldots, \lambda_d$ in the above expression, which is the permanent of $(g_{jk})$ divided by $\prod_{i=1}^{N} d_i!$.

It remainders to show that the permanent of M is the permanent of $(g_{jk})$ divided by $\prod_{i=1}^{N} d_i!$. Note that $M_{jk} \neq 0$ exactly when $g_{jk} \neq 0$, we inaugurate on N. For the base case, $d_1 = d$, and each nonzero entry of M is $\left(1/d!\right)^{1/d}$. A term in the permanent of M is the product of $d$ entries from M, so if it is nonzero it is $1/d!$. Thus the permanent of M is 1 = d! times the permanent of $1/d!$ as necessary. Now partition the matrix M and the matrix $(g_{jk})$ into two vertical bands conforming to the subsets $\bigcup_{i=1}^{N-1} T_i$ and $T_N$. The permanent can be calculated as the sum of a term for each choice of $d_N$ rows $1 \leq j_1 < \cdots < j_{d_N} \leq d$: compute the $(d - d_N) \times (d - d_N)$ sub permanent of the left band achieved by crossing out those rows, compute the $d_N \times d_N$ sub permanent of the right band corresponding to those rows, and multiply them composed. By the inductive hypothesis, the left sub permanent of M is the left sub permanent of $(g_{jk})$ divided by $\prod_{i=1}^{N} d_i!$ for the right sub permanent, every row is either all non-zero or all zero. If any row is all zero, both right sub permanents vanish. If every entry is nonzero, then all the entries are the same: $g_{jk} = 1$ and $M_{jk} = \frac{1}{(d_N!)^{1/d_N}}$. The right sub permanent of M is $d_N!\left(\left(1/d_N!\right)^{1/d_N}\right)^{d_N} = 1$ and the right sub permanent of $(g_{jk})$ is $d_N!$ so the whole term for M is the whole term for $(g_{jk})$ divided by $\prod_{i=1}^{N} d_i!$. We note that if the numbers are generic subject to the conditions given in Theorem 1, all the resolutions to the system will lie in the torus $(\mathbb{C}^*)^d$ in what follows we will refer to the number of resolutions in the torus $(\mathbb{C}^*)^d$ as the number of solutions by abuse of linguistic.

**Graphical games**

Kearns, Littman, and Singh well-defined the concept of graphical games, or games observing graphical models. (That paper deliberates purposeless graphs, but the postponement to focused graphs which we will use is straight onward.) A game among players $1, \dots, N$ obeys a directed graphical model, if the payoffs to player $i_1$ only depend on the activities of those players $i_1 \neq i_2$ for which there is an edge from $i_2$ to $i_1$ in the graphical model.

**Corollary 1**

Supposing a normal form game among players $i = 1, \dots, N$ with pure approach sets $S_i$ for each $i$ and effectiveness functions $u_i : \prod_{i \in I} S_i \to \mathbb{R}$ obeys a focused graphical model $Y$ with nodes $1, \dots, N$. Construct a graph G with nodes $\coprod_{i \in I} S_i$ such that there is an edge from $S_{ik}$ to $S_{jl}$ in G if and only if there is an edge from $i$ to $j$ in G. Then the system of equations denning the quasiequilibria of G satires the hypotheses of Theorem 1, so the number of such quasiequilibria in the generic case is given by the permanently formula.
For example, consider a game with 4 players, each with 3 pure approaches. Generally, such a game has quasiequilibria.

$$\operatorname{per}\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} = 297$$

In any case, assume now that amusement complies with a graphical model as in Figure 1. The hubs in the graphical model allude to the players, and the edges indicate that the result to the source player relies on upon the activities of the objective player. For quickness, compose a $a = \sigma_1(s_{11})$, $b = \sigma_1(s_{12})$, $c = \sigma_2(s_{21})$, $d = \sigma_2(s_{22})$, $e = \sigma_3(s_{31})$, $f = \sigma_3(s_{32})$, $g = \sigma_4(s_{41})$, $h = \sigma_4(s_{42})$. Since the payoff to player 1 depends only on the activities of player 2, connecting the payoff to player 1 from pure approaches $s_{10}$ and $s_{11}$ gives

$$\begin{aligned} u_1(s_{10}, s_{20}, \bullet)\sigma_2(s_{20}) + u_1(s_{10}, s_{21}, \bullet)\sigma_2(s_{21}) + u_1(s_{10}, s_{22}, \bullet)\sigma_2(s_{22}) \\ = u_1(s_{11}, s_{20}, \bullet)\sigma_2(s_{20}) + u_1(s_{11}, s_{21}, \bullet)\sigma_2(s_{21}) + u_1(s_{11}, s_{22}, \bullet)\sigma_2(s_{22}) \end{aligned}$$

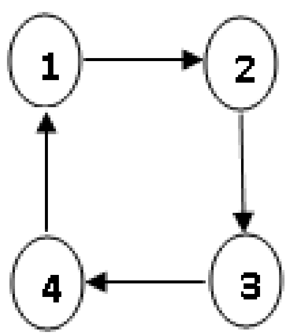


**Fig 1:** Graphical game

Or,

$$\begin{aligned} (u_1(s_{11}, s_{20}, \bullet) - u_1(s_{10}, s_{20}, \bullet))(1 - c - d) + (u_1(s_{11}, s_{21}, \bullet) - u_1(s_{10}, s_{21}, \bullet))c \\ + (u_1(s_{11}, s_{22}, \bullet) - u_1(s_{10}, s_{22}, \bullet))d = 0. \end{aligned}$$

Thus for player 1 we have two equations of the form

$$\bullet c + \bullet d + \bullet = 0,$$

For player 2 we have two equations of the form

$$\bullet e + \bullet f + \bullet = 0,$$

For player 3 we have two equations of the form

$$\bullet g + \bullet h + \bullet = 0,$$

And for player 4 we have two equations of the form

$$\bullet a + \bullet b + \bullet = 0.$$

Then the associated polynomial graph is described in Figure 2. The equation related

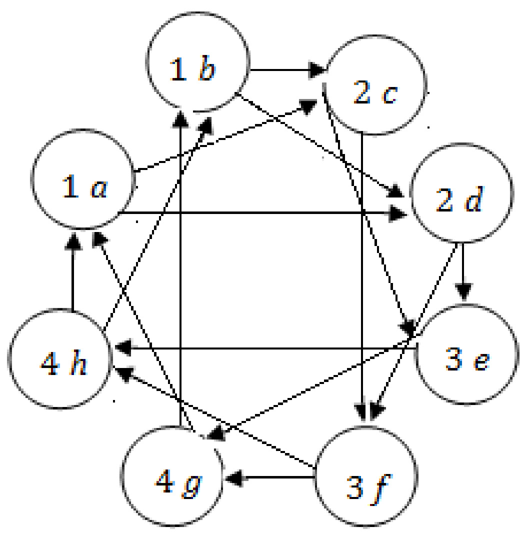


**Fig 2:** Associated polynomial graph for graphical game

With the node categorized 1*a* equates the payoffs to player 1 from selecting $s_{11}$ (which 1 does with possibility *a*) or choosing $s_{10}$ the game has Quasiequilibria.

$$\text{per}\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 1$$

Undoubtedly, this will dependably be the situation for a graphical model which is a coordinated cycle, where each player has the same number of unadulterated techniques. The reason is that the apathy conditions for this situation are straight, as we found in this case.

The polynomial graph G characterized in Theorem 1 contains more refined data than the graphical model. The segment into the $T_i$'s additionally can be more refined than the parcel of the arrangement of all unadulterated procedures into the arrangements of immaculate methodologies for each player. Next we will see a case of such a refinement while considering the diminishment of broad frame games to ordinary shape, where activities compare to branches of the game tree.

**Extensive-form games**
Now we consider extensive-form games. We begin by noting the following:

**Theorem 2-** All completely mixed Nash equilibria of an extensive form game are sub game picture-perfect.

**Proof:** Let $\sigma$ be a completely mixed Nash equilibrium of an extensive form game with N players definite by game tree T. Note that the approach prole induced by $\sigma$ on every sub game is also totally mixed. Let $\nu$ be a non-leaf node of T. Let $\tilde{\sigma}$ be the approach prole induced by $\sigma$ in the sub game induced by $\nu$. Let $\tilde{s}_j$ and $\tilde{t}_j$ be pure strategies of player $j$ in this sub game. Choose an action for $j$ at each node $\mu$ that is not a descendant of $\nu$ where $j$ acts, such that if $\mu$ is an ancestor of $\nu$ then $j$ chooses the branch important towards $\nu$, and use this select to extend $\tilde{s}_j$ and $\tilde{t}_j$ to pure policies $s_j$ and $t_j$ of player $j$ in the unique game. (So, $s_j$ and $t_j$ identify the same actions outside the sub tree.) Let $\nu_0 \cdots \nu_m = \nu$ be the single pathway from the root $\nu_0$ of T to $\nu$. We have $u_j(s_j, \sigma_{-j}) = u_j(t_j, \sigma_{-j})$. Let L be the set of all trees of T under $\nu$ and L' be the set of all other trees. Then

$$u_j(s_j,\sigma_{-j}) = \sum_{\lambda\in L} u_j(\lambda)\Pr[\lambda|(s_j,\sigma_{-j})] + \sum_{\lambda\in L'} u_j(\lambda)\Pr[\lambda|(s_j,\sigma_{-j})]$$
$$= \sum_{\lambda\in L} u_j(\lambda)\Pr[\lambda|(s_j,\sigma_{-j})] + \sum_{\lambda\in L'} u_j(\lambda)\Pr[\lambda|(t_j,\sigma_{-j})]$$

Since $s_j$ and $t_j$ choose the same actions outside the sub tree. Thus

$$\sum_{\lambda\in L} u_j(\lambda)\Pr[\lambda|(s_j,\sigma_{-j})] = \sum_{\lambda\in L} u_j(\lambda)\Pr[\lambda|(t_j,\sigma_{-j})] \qquad (1)$$

Furthermore, for any $\lambda \in L$, we have

$$\Pr[\lambda|(s_j,\sigma_{-j})] = \Pr[\lambda|(\tilde{s}_j,\tilde{\sigma}_{-j})]\prod_{k=0}^{m-1}\Pr[\nu_k \to \nu_{k+1}|(s_j,\sigma_{-j})]$$
$$= \Pr[\lambda|(\tilde{s}_j,\tilde{\sigma}_{-j})]\prod_{k=0}^{m-1}\Pr[\nu_k \to \nu_{k+1}|(t_j,\sigma_{-j})].$$

Observing that the collective element $\prod_{k=0}^{m-1}\Pr[\nu_k \to \nu_{k+1}|(t_j,\sigma_{-j})]$ in equation (1) is progressive by our choice of $s_j$, $t_j$ and because is fully mixed, we have that

$$u_j(\tilde{s}_j,\tilde{\sigma}_{-j}) = \sum_{\lambda\in L} u_j(\lambda)\Pr[\lambda|(\tilde{s}_j,\tilde{\sigma}_{-j})]$$
$$= \sum_{\lambda\in L} u_j(\lambda)\Pr[\lambda|(\tilde{t}_j,\tilde{\sigma}_{-j})]$$
$$= u_j(\tilde{t}_j,\tilde{\sigma}_{-j}).$$

Thus $\tilde{\sigma}$ is a (fully mixed) Nash equilibrium of the sub game induced by $\nu$.
In light of this perception, the partition and overcome way to deal with finding all Nash equilibria of an ordinary form game can be altered in the soul of in reverse acceptance to finding all sub game perfect equilibria (counting blended ones) of a broad form game. Review that in a typical form game, we would consider sub issues in which one immaculate procedure of one player $i$ was expelled. Currently we rather reflect sub matters in which, for some edge $\nu \to \mu$ where $i$ acts at $\nu$, we remove that edge and the perfect sub tree underneath $\mu$. We calculate the normal form for the game defined by this pruned tree and recursively and all its sub game perfect equilibria. Each such evenness $\sigma$ encourages equilibrium $\tilde{\sigma}$ in the sub game under $\nu$ in the cropped tree. To

check $\sigma$ is an equilibrium of the original game, we recursively calculate all the equilibria of the sub game under $\mu$ (where $i$ does not act), and check that for each such equilibrium $\tau$, we have $u_i(\tilde{\sigma}) \geq u_i(\tau)$.

We saw during the above proof that for a totally mixed approach prole $\sigma$, the equations $u_j(s_j, \sigma_{-j}) = u_j(t_j, \sigma_{-j})$ for all pure strategies $s_j, t_j$ of $j$ mean the relating conditions for each sub tree. The opposite suggestion additionally clearly holds.

We will now relate a polynomial graph to an arrangement of conditions for the quasiequilibria of a broad shape game, so that we can apply Theorem 1. For every hub in the game tree where a player acts, we will have a variable for each edge radiating from that hub aside from one recognized edge. This is on the grounds that the aggregate of the probabilities of picking each of those edges must be 1, so we wipe out one variable. Accordingly, we think about the settlements between picking the recognized edge and picking whatever other edge. The conditions will be lack of concern conditions for sub games of the broad form game.

All the monomials happening in these equations are four-sided free. For every sheet $\lambda$ under $\nu$, let the path from $\nu$, to $\lambda$ be $\nu = \nu_1 \dots \nu_k = \lambda$ then for any player $j$ with pure strategy $\tilde{s}_j$, we have $\Pr[\lambda|(\tilde{s}_j, \tilde{\sigma}_{-j})] = \prod_{l=1}^{k-1} \Pr[\nu_l \to \nu_{l+1}|(\tilde{s}_j, \tilde{\sigma}_{-j})]$, and every no continuous period in the product is $\sigma_n(\nu_l \to \nu_{l+1})$ for some player $n \neq j$, so for any edge *e* where n acts, the variable $\sigma_n(e)$ happens at most once in such a product. In fact $\sigma_n(e)$ occurs in such a product for at most one $e \in E(\nu_l)$. (That is, if $e, e' \in E(\nu_l)$ then $\sigma_n(e)$ and $\sigma_n(e')$ do not both occur in this monomial. So condition 2 of Theorem 9 holds.) When we eliminate $\sigma_n(e_{n\nu_l})$ we exchange it by an affine appearance, so this remainders true. Thus condition 1 of Theorem holds.

We now introduce an illustration where the arrangement of completely blended Nash equilibria is a positive-dimensional semi algebraic assortment. Consider the broad shape amusement indicated in Figure 3. The polynomial chart related with this game tree is delineated in Figure 4. For speediness, we comprise for occurrence $\sigma_1(C)$ for $\sigma_1(A \to C)$. The quasiequilibria conform a system of 4 equations as in Theorem. The equation connected with the edge $E \to G$ equates the payoff to player 3 from selecting this edge with that from selecting the edge $E \to F$, i.e, $u_3(F) = u_3(H)$. No variables follow in this equation, that is, it is an equation among constants. Similarly, the equation associated with the edge $E \to H$ is $u_3(F) = u_3(H)$, the equation connected with the edge $C \to E$ is $u_2(D) = u_2(E)$, where we have written $u_2(E)$ for the estimated payoff $u_2(E, \sigma_{-2})$ to player 2 for choosing the edge $C \to E$, given the approach prole of the other players. In this case

$$u_2(E) = u_2(F)\sigma_3(F) + u_2(G)\sigma_3(G) + u_2(H)\sigma_3(H), \text{ So}$$
$$u_2(D) = u_2(F) + (u_2(G) - u_2(F))\sigma_3(G) + (u_2(H) - u_2(F))\sigma_3(H).$$

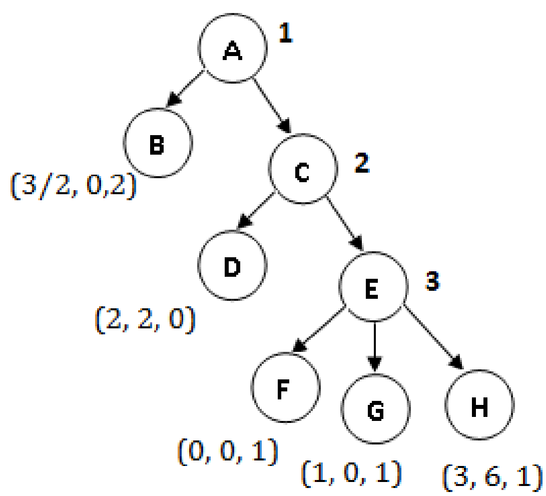


**Fig 3:** An Extensive Form Game

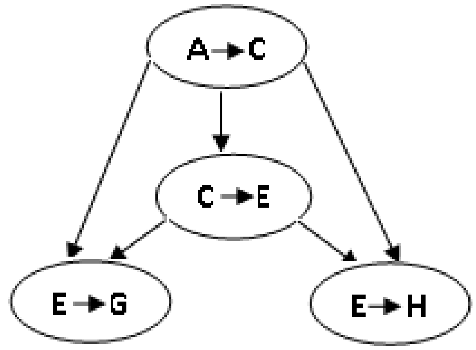


**Fig 4:** Associated Polynomial Graph for an Extensive Form Game

Finally, the equation associated to the edge $A \to C$ is

$$u_1(B) = u_1(C)$$
$$= u_1(D)(1 - \sigma_2(E) + u_1(F)\sigma_2(E)\big(1 - \sigma_3(G) - \sigma_3(H)\big) + u_1(G)\sigma_2(E)\sigma_3(G) + u_1(H)\sigma_2(E)\sigma_3(H)$$

Taking a gander at the specific adjustments in Figure 3, we see that the settlements to player 3 for picking F, G, or H are equivalent, as required. Likening the adjustments to player 2 for picking D or E, we get $6\sigma_3(H) = 2$ or $\sigma_3(H) = \frac{1}{3}$ these trees $\sigma_3(G)$ free to vary such that $0 < \sigma_3(G) < \frac{2}{3}$ finally, we must associate the payoffs to player 1 for selecting B or C. This gives

$$2(1 - \sigma_2(E) + \sigma_2(E)(\sigma_3(G) + 1) = \frac{3}{2}$$

Or,

$$\sigma_2(E)\big(1 - \sigma_3(G)\big) = \frac{1}{2}.$$

Thus the points $\sigma_3(G)$ and $\sigma_2(E)$ lie on a hyperbola. This hyperbola intersects the interior of the product of simplexes. For instance, the point $\sigma_3(G) = \frac{5}{12}$ (so $\sigma_3(F) = \frac{1}{4}$) and $\sigma_2(E) = \frac{6}{7}$ lies in this intersection. So the set of quasiequilibria is a portion of a hyperbolic cylinder, the product of a segment of a hyperbola with a line segment (since $\sigma_1(B)$ varies freely with $0 < \sigma_1(B) < 1$).

We can dissect this game a little further. Player 3 would like player 1 to here and there pick B, however can't force player 1 dependably to pick B, since if player 2 dependably picks D then both player 1 and player 2 are in an ideal situation with player 1 picking C. The best player 3 can do is make the adjustments to player 1 from picking B and C rise to. Presently if player 3 made player 2 get a more prominent result from picking D than E, then player 2 would dependably pick D, player 1 would dependably pick C, and player 3 would get nothing. So player 3 must make $u_2(D) \le u_2(E)$ we analyzed the case $u_2(D) = u_2(E)$ above. If player 3 make $\sigma_3(H) > \frac{1}{3}$, then $u_2(D) < u_2(E)$ and player 2 will always select E. Then the payoff to player 1 from choosing $\sigma_3(G) + 3\sigma_3(H)$, thus we have $\sigma_3(G) + 3\sigma_3(H) = \frac{3}{2}$ with $\frac{1}{3} < \sigma_3(H) \le \frac{1}{2}$ (this makes $0 \le \sigma_3(G) < \frac{1}{2}$ and $\frac{1}{6} < \sigma_3(F) \le \frac{1}{2}$). Then $\sigma_1(C)$ varies freely with $0 \le \sigma_1(C) \le 1$, so we have a rectangle of mostly blended equilibria. Player 3 is in an ideal situation picking these, from that point forward the result D where player 3 gets zero adjustments is never come to. Along the line $\sigma_3(G) + 3\sigma_3(H) = \frac{3}{2}$, equilibria with superior $\sigma_3(H)$ Pareto dominate those with smaller $\sigma_3(H)$, i.e., they make some player improved off and no player inferior off. Particularly, the payoff to player 2 growths, the payoff to player 1 is always $\frac{3}{2}$, and the payoff to player 3 stays the same at $2\big(1 - \sigma_1(C)\big) + \sigma_1(C) = 2 - \sigma_1(C)$. Thus the Pareto dominant equilibrium between those on this line is that player 3 has $\sigma_3(F) = \frac{1}{2}$, $\sigma_3(G) = 0$ and $\sigma_3(H) = \frac{1}{2}$. On the other hand, at the immaculate technique balance where player 3 dependably picks H, we have that player 1 dependably picks C, and the result to player 3 tumbles from $2 - \sigma_1(C)$ to 1, hush player 3 does not lean toward this balance, and rather blends F and H similarly to have some shot of a higher payoff. As $\sigma_1(C)$ increases, the result to player 3 diminishes and the result to player 2 expands, so the equilibria along this line don't Pare to overwhelm each other. In this manner without presenting different issues, (for example, hazard avoidance) there is no basis for foreseeing which of the equilibria along the line $0 < \sigma_1(\mathrm{C}) < 1$, $\sigma_2(\mathrm{E}) = 1$, $\sigma_3(\mathrm{F}) = \sigma_3(\mathrm{H}) = \frac{1}{2}$ should be chosen.

## Conclusion

Game theory is a mathematical model of strategic interaction. The main computer package for studying game theory today is Gambit. Although there are many ways to characterize Nash equilibria, the one which lends itself most easily to the computation of all Nash equilibria of a game with more than two players is a solution to systems of polynomial equations. However, the algorithm currently implemented in Gambit could be out performed by the existing polyhedral homotopy continuation software PHC. So hopefully PHC or some similar package will soon be incorporated into Gambit. Furthermore, there are many other promising directions to pursue in applying algebra, and in particular computer algebra, to game theory.

Furthermore, we could have taken a voyage into the universe of games with draws. As specified, Fraenkel has done some work on such games. He has dissected the multifaceted nature of numerous games, which is essential when fathoming these combinatorial games on PCs. Another street we could had taken was to contrast this kind of game hypothesis with different sorts of game theory, e.g. to the prudent game theory that sprung from the work of von Neumann and Morgenstern.

## Future research

Algebraic method in game theory is a field that has gotten little consideration throughout the year; its theory is very broad. Be that as it may, just little research has been done on its applications in the fields of computerized reasoning. Thus, there are as yet numerous ranges that can be investigated. Moreover, for all-small games (i.e. diversion for which the combinatorial game-theoretic values just comprise of infinitesimals) it is conceivable to figure the nuclear weight of a position, additionally called its snootiness, since it is the number of ups the estimation of the position is in all likelihood equivalents.

## References


1. Michael Kearns, Michael Littman L, Satinder Singh. Graphical models for game theory. Uncertainty in artificial intelligence: proceeding of the seventeenth conference, 2001, 253-260.
2. Selman Akbulut, Henry King C. Topology of real algebraic sets. Springer Verlag, 1992.
3. David Cox, John Little, DonalO' Shea. Ideals, Varieties, and Algorithms. Springer Verlag, 1997.
4. Ruchira Datta S. Using computer algebra to compute Nash equilibria. Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation, 2003, 7479.
5. IoannisEmiris and John Canny. Efcient incremental algorithms for the sparse resultant and the mixed volume. Journal of Symbolic Computation, 1995; 20(2):117149.
6. Jean-Charles Faugere. A new efcient algorithm for computing Groebner- bases without reduction to zero (F5). In Proceedings of the 2002 International Symposium on Symbolic and Algebraic Computation (ISSAC 2002), 2002; 75:83.
7. Takayuki Gunji, Sunyoung Kim, Masakasu Kojima, Akiko Takeda, Katsuki Fujisawa, Tomohiko Mizutani. PHoMa polyhedral homotopy continuation method for polynomial systems. Research Report B-386, Tokyo Institute of Technology, Tokyo, 2003. http://www.is.titech.ac.jp/kojima/PHoM/.
8. Andrew McLennan. The maximum number of real roots of a multihomogeneous system of polynomial equations. Beitragezur Algebra und Geometrie, 1999; 40(2):343350.
9. Richard McKelvey, Andrew McLennan. Computation of equilibria in nitegames.In H. Amman, D. A. Kendrick, and J. Rust, editors, The Handbook of Computational Economics, 1996; I:87142. Elsevier.
10. Richard McKelvey, Andrew McLennan. The maximal number of regular totally mixed Nash equilibria. Journal of Economic Theory, 1997; 72:411 425.